KCS-301 DISCRETE STRUCTURES & THEORY OF LOGIC (DSTL)

① Literal meaning.

DISCRETE :- → NonContinuous. [Ex: Discrete Sets ^collection includes finite and Countable sets]

~~MATHEMS~~
STRUCTURES/MATHEMATICS:- form; Ex of structures that are discrete are Combinations, graphs, and logical statements.

THEORY OF LOGIC:- → The sensible reason or way of thinking.
[an idea or set of ideas that tries to explain something]

② why Study.

⇒ It is a vital prerequisite to learning algorithms, at it covers probabilities, trees, graphs, logic, mathematical thinking and much more. It simply explains them, so once you get these basic topics, it is easier to dig into algorithms.

⇒ The problem-solving techniques honed in a discrete mathematics are necessary for writing complicated software. Students who are successful in discrete mathematics will be able to generalize from a single instance of a problem to an entire class of problems, and to identify and abstract patterns from data.

⇒

"It's a building block for logical thinking."

③ Real life Example:- "wiring a Computer network using the least amount of cable is a minimum-weight spanning tree problem."

④ Discrete vs Continuous

⇒ Discrete data is the type of data that has clear spaces between values. continuous data is data that falls in a constant sequence.

⇒. Discrete data is countable, while continuous - measurable.

Ex:- Discrete
(i) The number of students in a class.
(ii) The number of workers in a Company
(iii) The number of parts damaged during transportation
(iv) Shoe Sizes.
(v) Number of languages an individual speaks.
(vi) The number of test questions you answered correctly.

- The amount of time required to complete a project.
- The height of children.
- The amount of rain, in inches, that falls in a storm
- The square footage of a two-bedroom house.
- The weight of a truck.
- The speed of cars

SET:- Group of things that belong together.

(5) CO'S Course OutComes.

CO1:- Write an argument using logical notation and determine if the argument is or is not Valid.

CO2:- Understand the basic principles of sets and operations in sets

CO3:- Demonstrate an understanding of relations and functions and ~~to~~ be able to determine their properties.

CO4:- Demonstrate the different traversal methods for trees and graphs.

CO5:- Model problems in Computer Science using graphs and trees.

(6) Syllabus.

- Theory (5 units)
- Lab. (Programs in 'C' ~~program~~ and ~~of~~ mapple Software)

5 units.
- UNIT 1:- Set Theory, Functions, Natural Numbers.
- UNIT 2:- Algebraic structures
- UNIT 3:- Lattices, Boolean Algebra.
- UNIT 4:- Propositional Logic, Predicate Logic.
- UNIT 5:- Trees, graphs and Combinatorics.

Planning of Lecture:-

→ Greetings to Student. (wish + Self name & Subject Name)
→ ~~Literal meaning of Sub~~
→ Student's Introduction.
→ Motivation to move forward Now.
⇒ Literal meaning of subject. + why study + COS + syllabus
→ SET

Student's Introduction.

Common Question

i) Your Name:-
ii) meaning of Name
iii) Who are you?
~~How are you~~
iv) What is the significance of your Name in your life?

DSTL

UNIT-1

Set Theory :- Introduction, Combination of sets, Multisets, ordered pairs. Proofs of some general Identies on sets

Relations: Definition, operations on relations, properties of relations, Composite Relations, Equality of relations, recursive definition of relation, order of Relations.

Functions:- Definition, classification of functions, operations on functions, Recursively defined functions, Growth of functions.

Natural Numbers:- Introduction, Mathematical Induction, Variants of Induction, Induction with Nonzero Base Case, Proof methods, Proof by Counter - example, Proof by Contradiction.

Lecture 1.

Introduction

SETS and Elements

(i) → A set is a collection of objects, the elements or members of the set.

Sets denotion :- A, B, X, Y, - - - (capital letters)

elenets " :- a, b, x, y, - - - (~~Small letters~~ lower Case)

(ii) $p \in A$ :: 'p belongs to A' or 'p is an element of A'

$p \notin A$ :: 'p does not belong to A' or 'p is not an element of A'

(iii) Principle of extension:- 'Two sets A and B are equal if they have the same members'.

Ex:- (i) $A = \{1, 2, 4, 7\}$, $B = \{4, 2, 1, 7\}$

$A = B$ ✓

(ii) $A = \{1, 2, 4, 7\}$, $B = \{1, 2, 3, 7\}$

$A \neq B$

(iii) **Specifying Sets :- Two ways.**

(i) $A = \{a, e, i, o, u\}$ — A set containing element.

(ii) $A = \{x : x$ ~~is a vowel of English~~ is a letter in the English alphabet, x is a vowel$\}$ — State those properties which characterized the elements in the set.

$[b \notin A, e \in A]$

Ex: $B = \{x : x \text{ is an even integer}, x > 0\}$

$\Longrightarrow$ "B is the set of x such that x is an even integer and x is greater than 0". → denotes the set 'B' whose elements are positive integers.

Ex: $E = \{x : x^2 - 3x + 2 = 0\}$

↳ E is a solution set of given equation.

so $E = \{1, 2\}$

Ex: $F = \{2, 1\}$

$G = \{1, 2, 2, 1, \frac{6}{3}\}$, then $E = F = G$.

[A set remains the same if it's elements are repeated or rearranged] property

Standard Notations.

N = the set of positive integers

Z = the set of integers

Q = the set of rational numbers $\{[a/b]$ a, b are integers$\}$

R = the set of real numbers

C = the set of complex numbers. $[2+3i]$

(iv) **Principle of Abstraction:-** "Given any set U and any property P, there is a set A such that the elements of A are exactly those members of U which have the property P." → [~~describing a set~~ used for]

(v) Universal set and Emptyset.

→ In any application of the theory of sets, the members of all sets under investigation usually belong to some fixed large set called the universal set:

for Ex:- In plane geometry the universal set consists of all the points in the ~~pt~~ plane.

Ex:- In Human population studies the universal set consists of all people in the world.

Symbol: $\boxed{U}$

Emptyset:- For a given set U and a property P, there may not be any elements of U which have property P. ~~For Ex~~:

Ex:- $S = \{x : x \text{ is a positive integer}, x^2 = 3\}$

has no elements since no positive integer has the required property.

⇒ The set with no elements is called the emptyset, or null set and is denoted by ϕ.

$S = \{\}$
$S = \phi$ > Same representation.

$T = \{\phi\}$ → A set is having an empty set.

$S \neq T$

(vi) SUBSETS:- If every element in a set A is also an element of a set B, then A is called a subset of B.

$A \subseteq B$ or $B \supseteq A$

"A is contained in B" or "B contains A".

$A \nsubseteq B$ or $B \nsupseteq A$ ⇒ A is not a subset of B.
⇒ at least one element of A does not belong to B.

Ex:- $A = \{1, 3, 4, 5, 8, 9\}$, $B = \{1, 2, 3, 5, 7\}$, $C = \{1, 5\}$

Imp Relation

$$\boxed{N \subseteq Z \subseteq Q \subseteq R}$$

→ $E = \{2, 4, 6\}$, $F = \{6, 2, 4\}$

$\boxed{E \subseteq F}$. and $\boxed{F \subseteq E}$

$\Rightarrow \boxed{E = F}$ ✓

* ⇒ Every set is a subset of itself. ✓ (since, trivially, elements of A belongs to element of A).

* ⇒ Every set A is ~~...~~ a subset of ~~of~~ the universal set U since, ~~$A \subseteq U$~~ since all the elements of A belong to U.

⇒ The empty set ϕ is a subset of A.

$\{\ \}$ $\{\text{---},\ \}$

→ $A \subseteq B$ {every element of A belong to B}
$B \subseteq C$ {every element of B belong to C}

$\Rightarrow \underline{A \subseteq C}$ {since every element of A belong to C}

<u>Theorem</u> (i) for any set A, we have $\phi \subseteq A \subseteq U$

(ii) For any set A, we have $A \subseteq A$

(iii) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$

(iv) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$

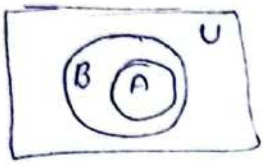
* ⇒ If $A \subseteq B$, then it is still possible that $A = B$. When $A \subseteq B$ then but $A \neq B$, we say A is a <u>proper subset of</u> B.

$\Rightarrow \underline{A \subset B}$

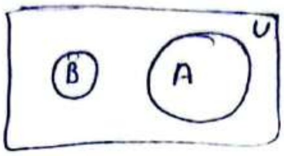
<u>Ex:</u> $A = \{1, 3\}$ $B = \{1, 2, 3\}$, $C = \{1, 3, 2\}$

⇒ A and B are both ~~...~~ subsets of C; but A is a proper subset of C, whereas B is not a proper subset of C since $B = C$.

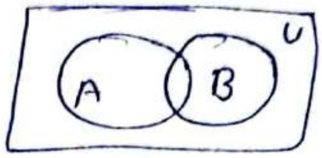
VENN DIAGRAM :- Pictorial respresentation of sets in which sets represeted by enclosed area in the plane.



(a) A C B



(b) A and B are Disjoint



(C)

→ some objects are in A, but not in B.
→ some are in B but not in A.
→ an some are in both
→ Some are in neither A nor in B

Arguments and Venn diagrams

Ex:-

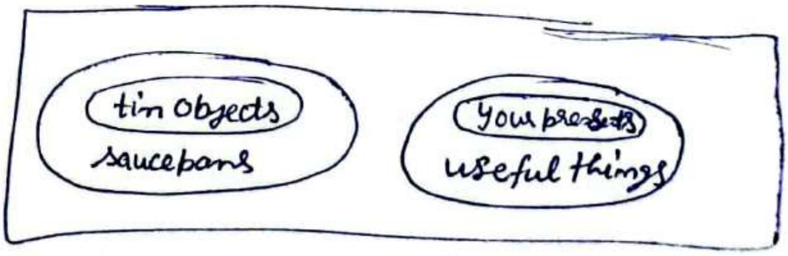
S1: my saucepans are the only things ~~th~~ I have that are made of tin.
S2: I find all your presents very useful.
S3: None of my saucepans is of the slightest use.

S: Your presents to me are not made of tin

S_1, S_2, S_3 are assumptions

S ⇒ Conclusion

⇒ ~~By S~~



By S_1; the ~~time~~ tin objects are contained in the set of saucepans and by S_3 the set ~~set~~ of saucepans and the set of useful things are disjoint;

By S_2 the set of "your presents" is a subset of the set of useful things.

~~hence draw fig~~.

⇒ The Conclusion is clearly valid by the above venn diagram because the set of 'your presents' is disjoint from the set of tin objects.

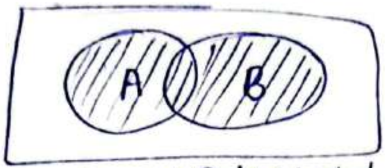
SET OPERATIONS

(1) Union and Intersection

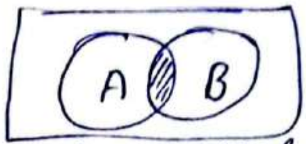
⇒ The union of two sets A and B, denoted by $A \cup B$, is the set of all elements which belong to A or to B; that is

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

Here "or" is used in the sense and/or,



(a) $A \cup B$ is shaded



(b) $A \cap B$ is shaded.

The intersection of two sets A and B, denoted by $A \cap B$, is the set of elements which belong to both A and B; that is

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

Ex:- Let $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6, 7\}$, $C = \{2, 3, 5, 7\}$

$A \cup B = \{$ ___

$A \cap B = \{$ ___

$A \cup C = \{$ ___

$A \cap C = \{$ ___

Ex:- ~~Let J contains the set of AIML-1 students,~~

Let M denotes the set of male students in GL Bajaj, and let F denotes.

$$M \cup F = C$$

$$M \cap F = \phi.$$

Theorem:- The following are equivalent: $A \subseteq B$, $A \cap B = A$, and $A \cup B = B$

(2) Complirents :

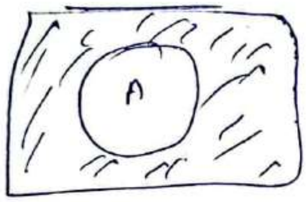
A^c, is the set of elemets which belong to U but do not belong to A ; that is

$$A^c = \{x : x \in U, x \notin A\}$$

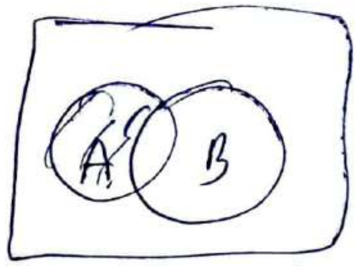
{ Other Denotion $A^c = A' = \bar{A}$ }

⇒ The relative Complement of a set B with respect to a set A or, simply, the difference of A and B, denoted by $A \setminus B$, is the set of elemets which belong to A but which do not belong to B ; that is

$$A \setminus B = \{x : x \in A, x \notin B\}$$



A^c is shaded.



$A \setminus B$ is shaded

Ex:- $U = N = \{1, 2, 3. ...\}$, the positive integers, is the universal set. , $E = \{2, 4, 6, 8. ...\}$ → even integers.
$A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6, 7\}$, $C = \{6, 7, 8, 9\}$ → A^c, B^c, C^c → $A \setminus B, B \setminus C, B \setminus A, C \setminus E$

(4) Symmetric Difference.

$$A \oplus B = (A \cup B) \setminus (A \cap B)$$

or

$$A \oplus B = (A \setminus B) \cup (B \setminus A)$$

$$A = \{1, 2, 3, 4, 5, 6\} \text{ and } B = \{4, 5, 6, 7, 8, 9\}$$

(5) Fundamental Products :- Consider n distinct sets $A_1, A_2, --- A_n$. A fundamental product of the sets is a set of the form

$$A_1^* \cap A_2^* \cap ... \cap A_n^*$$

where A_i^* is either A_i or A_i^c.

(1). There are 2^n such fundamental products.
(2) any two such fundamental products are disjoints, and
(3) the universal set U is the union to all the fundamental products.

Ex: ~~U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}~~

A = {1, 2, 3}

B = {6, 7, 8}

C = {4, 5, 9}

Algebra of Sets and DUALITY

Sets under the operations of union, interaction, and complement satisfy various laws or identities which are listed below.

$\boxed{(A \cup B)^c = A^c \cap B^c} \longrightarrow$ DeMorgan's laws

(1) Idempotent laws

(1a) $A \cup A = A$ (1b) $A \cap A = A$

(2) Associative laws

(2a) $(A \cup B) \cup C = A \cup (B \cup C)$ (2b) $(A \cap B) \cap C = A \cap (B \cap C)$

(3) Commutative laws

(3a) $A \cup B = B \cup A$ (3b) $A \cap B = B \cap A$

(4) Distributive laws.

(4a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (4b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(5) Identity Laws

(5a) $A \cup \phi = A$ 5(b) $A \cap U = A$

(6a) $A \cup U = U$ 6(b) $A \cap \phi = \phi$

Involution laws

(7) $(A^c)^c = A$

Complement Laws.

8(a) $A \cup A^c = U$ 8(b) $A \cap A^c = \phi$

9(a) $U^c = \phi$ (9b) $\phi^c = U$

DeMorgan's Laws

(10a). $(A \cup B)^c = A^c \cap B^c$ (10b) $(A \cap B)^c = A^c \cup B^c$

\# Principle of Duality.

Suppose E is an equation of set algebra, The dual E^* of E is the equation obtained by replacing each occurance of $\cup, \cap, U$, and ϕ in E by $\cap, \cup, \phi$, and U respectively.

$\cup$ — union, U — universal set; $\cup$ — union, U — universal set

E

for Ex:- the dual of.

$(U \cap A) \cup (B \cap A) = A$ is $(\phi \cup A) \cap (B \cup A) = A$

→ This fact (concept) is known as principle of duality.

If any equation E is an identity, then its dual E^* is also an identity.

Key Points (i) If A and B are disjoint finite sets, then $A \cup B$ is finite and

$$\boxed{n(A \cup B) = n(A) + n(B)}$$

(ii) If A and B are finite sets, then $A \cup B$ and $A \cap B$ are finite and.

$$\boxed{n(A \cup B) = n(A) + n(B) - n(A \cap B)}$$

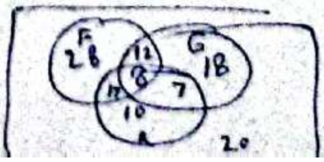
(iii) If A, B, and C are finite sets, then so is $A \cup B \cup C$, and

$$\boxed{n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)}$$

Ex:- Consider the ~~dat~~ fallowig data for 120 Mathematics students at a college concerning the languages French, German, and Russian:

65 study French
45 study German, 42 study Russian.
20 study french and German
25 study French and Russian
15 study german and Russian.
8 study all three languages.

How many students study only one of the languages?



Classes of Sets, Power Sets & Partitions

(Source : Tutorialspoint.com)

(Power set

$P(S) \rightarrow$ set of all possible subsets of $S = \{a, b, c, d$

Multisets

A multiset is an unordered collection of elements, in which the multiplicity of an element may be one or more than one or zero. The multiplicity of an element is the number of times the element repeated in the multiset.

$A = \{l, l, m, m, n, n, n, n\}$

$B = \{a, a, a, a, a, c\}$

Operations.

(i) Union of multisets.

$A = \{l, l, m, m, n, n, n\}$

$B = \{l, m, m, m, n\}$

$A \cup B = \{l, l, m, m, m, n, n, n, n\}$

$\Rightarrow$ the multiplicity of an element is equal to the maximum of the multiplicity of an element in A and B.

(ii) Intersection of multisets

$A = \{l, l, m, n, p, q, q, r\}$

$B = \{l, m, m, p, q, r, r, r, r\}$

$A \cap B = \{l, m, p, q, r\}$.

the multiplicity of an element is equal to the minimum of the multiplicity of an element in A and B.

(iii) Difference of multisets

$A = \{l, m, m, m, n, n, n, p, p, p\}$

$B = \{l, m, m, m, n, r, r, r\}$

$A - B = \{n, n, p, p, p\}$

the multiplicity of an element is equal to the multiplicity of the element in A minus the multiplicity of the element in B if the difference is +ve, and is equal to zero if the difference is 0 or negative)

(iv) Sum of Multisets: The sum of two multisets A and B, is a multiset such that the multiplicity of an element is equal to the sum of the multiplicity of an element in A and B.

$A = \{l, m, n, p, r\}$

$B = \{l, l, m, n, n, n, p, r, r\}$

$A+B = \{l, l, l, m, m, n, n, n, n, p, p, r, r, r\}$

(v) Cardinality of sets: The no. of distinct element in a multiset without considering the multiplicity of an element.

$A = \{l, l, m, m, n, n, n, p, p, p, p, q, q, q\}$

Cardinality of set A is 5.

ordered set: ⇒ ordered collection of distinct objects.

~~It is de~~

Roll no: $\{3, 6, 7, 8, 9\}$

week days = $\{S, M, T, W, W, TH, F, S, S\}$

(a, b) and (b, a) are two different ordered pair.

Ist element as ordered pair

$\{(a, b), c\}$ → ordered triple

Ist element an ordered triple.

Ordered Quadable. $\{(((a, b), c), d)\}$

ex: ordered set of 5 elements

$\{(((\underbrace{(a, b), c), d}_{(n-1)})\ \underbrace{e}_{5^{th}}\}$

Relations

Relations will be defined as ~~as~~ in terms of ordered pairs (a,b) of elements, where a is designated as first element and b as the second element.

→ $(a,b) = (c,d)$ if and only if $a = c$ and $b = d$.

Thus $(a,b) \neq (b,a)$ unless $a = b$.

Product Sets

Two set A, B. The set of all ordered pairs (a,b) where $a \in A$ and $b \in B$ is called the product, or Cartesian product, of A and B.

$$\boxed{A \times B = \{(a,b) : a \in A \text{ and } b \in B\}}$$

for $A \times A$, can also be written as A^2.

Ex: R denotes the real numbers. so $R^2 = R \times R$, is the set of ordered pairs of real numbers.

Ex: $A = \{1,2\}$, $B = \{a,b,c\}$ then

$A \times B = \{(1,a), (1,b), (1,c), (2,a), (2,b), (2,c)\}$

$B \times A = \{(a,1), (a,2), (b,1), (b,2), (c,1), (c,2)\}$

Also $A \times A = \{(1,1), (1,2), (2,1), (2,2)\}$

$$A \times B \neq B \times A$$

$$\boxed{n(A \times B) = 6 = 2 \cdot 3 = n(A) \cdot n(B)}$$

* Relations

Def^n : Let A and B be sets. A binary relation or, simply, relation from A to B is a subset of $A \times B$.

(i) $(a, b) \in R$; and we then say "a is R-related to b", written aR_b

(ii) $(a, b) \notin R$; we say "a is not R-related to b", written $a \not R_b$

$A^2 = A \times A \Rightarrow R$ is a relation on A.

* The domain of a relation R is the set of all first elements of the ordered pairs which belong to R, and the range of R is the set of 2nd elements.

Ex:-

(a) $A = \{1, 2, 3\}$ and $B = \{x, y, z\}$

and let $R = \{(1, y), (1, z), (3, y)\}$

then R is a relation from A to B since R is a subset of $A \times B$.

$1R_y$, $1R_z$, $3R_y$, but $1 \not R_x$, $2 \not R_x$, $3 \not R_z$, $3 \not R_x$, $2 \not R_z$, $2 \not R_y$.

(b) A familiar relation on the set Z of integers is "m divides n". A common notation for this relation is to write m/n when m divides n. Thus $6|30$ but $7 \nmid 25$.

(c). Consider the set L of lines in the plane. Perpendicularity written $\perp$, is a relation on L. That is given any pair of lines a and b, either $a \perp b$ or $a \not\perp b$.

(d). Let A be any set. Then $A \times A$ and ϕ are subsets of $A \times A$ hence are relations on A called the universal relation or empty relation.

Inverse Relation:-

$\Rightarrow$ Let R be any relation from a set A to a set B. The Inverse of R, denoted by R^{-1}, is the relation from B to A which consists of those ordered pairs, which when reversed, belong to R; that is,

$$R^{-1} = \{ (b,a) : (a,b) \in R \}$$

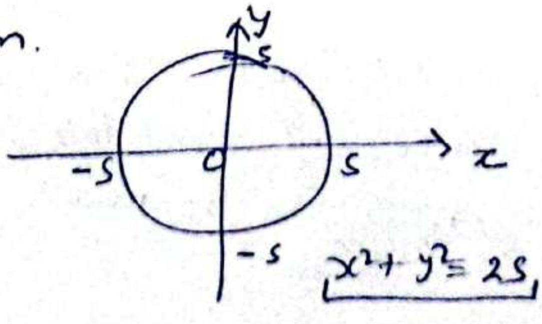
Ex let $R = \{(1,y),(1,z),(3,y)\}$ from $A = \{1,2,3\}$
$B = \{x,y,z\}$

$$\boxed{R^{-1} = \{(y,1),(z,1),(y,3)\}}$$

$$\boxed{(R^{-1})^{-1} = R}$$

Pictorial Representation:- $\underbrace{x^2+y^2=25}_{}$
relation S.

$\Rightarrow$ S consists of all ordered pairs (x,y) which satisfy the given equation.

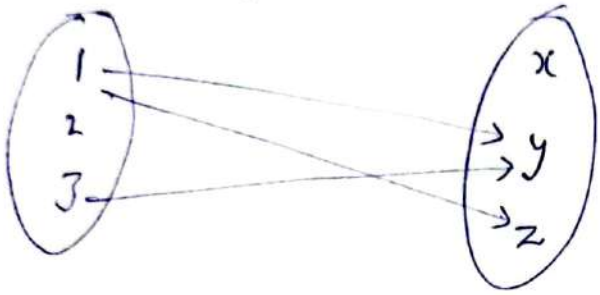


Representation of Relations on finite sets. (Two ways)

(i)

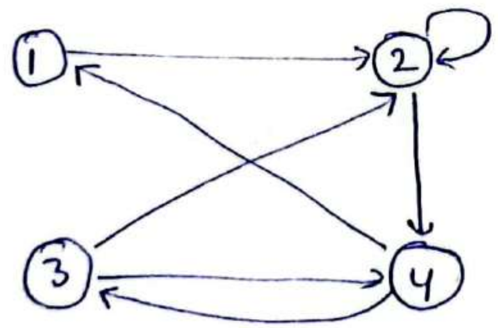
	x	y	z
1	0	1	1
2	0	0	0
3	0	1	0

(ii)



Directed graph of Relations on sets :

$R = \{(1,2), (2,2), (2,4), (3,2), (3,4), (4,1), (4,3)\}$



Composition of Relations

A, B, C are sets, R & S are relations.

$_A R_B$, $_B S_C$ $\Rightarrow$ R is a subset of $A \times B$.

$\Rightarrow$ S is a subset of $B \times C$

then R and S give rise to a relation from A to C denoted by $R_o S$ and defined by

$a(R_o S)_c$ if for some $b \in B$ we have $a R_b$ and $_b R_c$.

That is $R_o S = \{(a,c) :$ there exist $b \in B$ for which $(a,b) \in R$ and $(b,c) \in S\}$

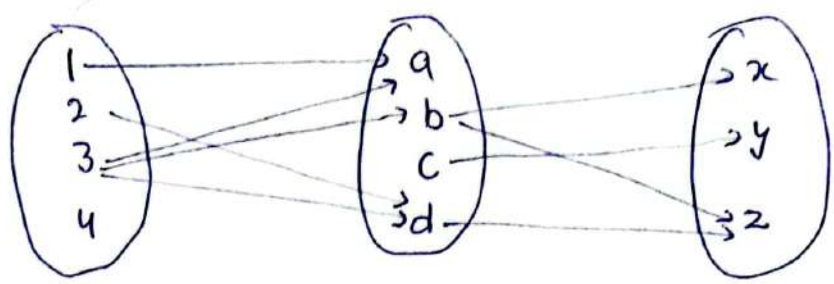
⇒ Composition of R with itself is always defined.

$R \circ R \cong R^2$

→ $R^3 = R \circ R \circ R = R^2 \circ R$ and so on. R^n is defined for all +ve n.

Ex: Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c, d\}$, $C = \{x, y, z\}$ and let $R = \{(1,a), (2,d), (3,a), (3,b), (3,d)\}$ and $S = \{(b,x), (b,z), (c,y), (d,z)\}$

↓
Diagram



✓ $2 (R \circ S)_z$ since $2 R_d$ and $d S_z$

✓ ~~3 (R∘S)x and 3~~

$R \circ S = \{(2,z), (3,x), (3,z)\}$ ✓

Types of Relations.

(i) Reflexive Relations :- A relation R on a set A is reflexive if aRa for every $a \in A$, that is, if $(a,a) \in R$ for every $a \in A$. Thus R is not reflexive if there exist an $a \in A$ such that $(a,a) \notin R$.

Ex: $A = \{1,2,3,4\}$ — which one is reflexive

$R_1 = \{(1,1), (1,2), (2,3), (1,3), (4,4)\}$

$R_2 = \{(1,1), (1,2), (2,1), (2,2), (3,3), (4,4)\}$

$R_3 = \{(1,3), (2,1)\}$ $R_4 = \phi$, empty relation

$R_5 = A \times A$, Universal ...

Example 1

Consider the following relations on set $A = \{1,2,3,4\}$

Relation	Reflexive	Symmetric	Antisymmetric	Transitive	Equivalence
$R_1 = \{(1,1),(1,2),(2,3),(1,3),(4,4)\}$	×	×	✓	✓	
$R_2 = \{(1,1),(1,2),(2,1),(2,2),(3,3),(4,4)\}$	✓	✓	×	✓	
$R_3 = \{(1,3),(2,1)\}$	×	×	✓	×	
$R_4 = \emptyset$ the empty relation	×	✓	✓	✓	
$R_5 = A \times A$ the universal relation	✓	✓	×	✓	

Ex 2:- Consider the following five relations.

Relation	Ref.	Sym.	Antisym.	Transitive	Equiv.
(i) Relation $\le$ (less than or equal) on the set Z of integers.	✓	×	✓	✓	
(ii) set inclusion $\subseteq$ on a collection C of sets	✓	×	✓	✓	
(iii) Relation $\perp$ (perpendicular) on the set L of lines in the plane.	×	✓	×	×	
(iv) Relation $\parallel$ (parallel) on the set L of lines in the plane	×	✓	×	×	
(v) Relation $\mid$ of divisibility on the N of positive integers	✓	×	✓	✓	

Types of Relations.

(A). Reflexive :- A relation R on a set A is reflexive if aRa for every $a \in A$, that is, if $(a,a) \in R$ for every $a \in A$. Thus R is not reflexive if there exists an $a \in A$ such that $(a,a) \notin R$.

(B) Symmetric:- A relation R on a set A, is symmetric if whenever aRb then bRa, that is, if whenever $(a,b) \in R$ then $(b,a) \in R$. Thus R is not symmetric if there exists ~~(a,b)~~ $a,b \in A$ such that $(a,b) \in R$ but $(b,a) \notin R$.

(C) <u>Antisymmetric</u>:- A <u>relation R</u> on a <u>set A</u> is antisymmetric if whenever aR_b and bR_a then $a=b$, that is, if whenever $(a,b), (b,a) \in R$ then $a=b$. Thus R is not antisymmetric if there exists $(a,b) \in A$ such that (a,b) and (b,a) belong to R, but $a \neq b$.

<u>Remark</u>:- The properties of being symmetric and being antisymmetric are not negatives of each other. For example, the relation R $R = \{(1,3), (3,1), (2,3)\}$ is neither symmetric nor antisymmetric. On the other hand, the relation $R' = \{(1,1), (2,2)\}$ is both symmetric and antisymmetric.

(D) <u>Transitive</u>:- A relation R on a set A is transitive if whenever aR_b and bR_c then aR_c, that is, if whenever $(a,b), (b,c) \in R$ then $(a,c) \in R$. Thus R is not transitive if there exists $a,b,c \in A$ such that $(a,b), (b,c) \in R$ but $(a,c) \notin R$.

⇒ $\perp$ is not transitive. If $a \perp b$ and $b \perp c$, then it is not true that $a \perp c$. Since no line is parallel to itself, we can have $a \parallel b$ and $b \parallel a$, but $a \nparallel a$. Thus $\parallel$ is not transitive.

(E) <u>Equivalence Relations</u>:- Consider a nonempty <u>set A</u>. A <u>relation R on A</u> is an equivalence relation if R is reflexive, symmetric and transitive. That is, R is an equivalence relation on A if it has the following three properties.

(i) for every $a \in A$
(ii) if aRb, then bRa
(iii) if aR_b and $_bR_c$, then aR_c

for example, alike objects.
(i) $a=a$ for all $a \in A$
(ii) If $a=b$, then $b=a$
(iii) If $a=b$, and $b=c$, then $a=c$.

$| - |$ $(a,b), (b,a)$

1.1.1

Closure Properties.

Reflexive and Symmetric closure

→ Here $\Delta_A = \{(a,a), a \in A\}$ is the diagonal or equality relation on A.

Thⁿ: Let R be a relation on set A. Then:

(i) $R \cup \Delta_A$ is the reflexive closure of R.

(ii) $R \cup R^{-1}$ is the symmetric closure of R.

In other words, reflexive(R) is obtained by simply adding to R those elements (a,a) in the diagonal which do not already belong to R, and symmetric (R) is obtained by adding to R all pairs (b,a) whenever belong to R.

Ex: $A = \{1,2,3,4\}$

$R = \{(1,1),(1,3),(2,4),(3,1),(3,3),(4,3)\}$

Then, reflexive(R) $= R \cup \{(2,2),(4,4)\}$ and

symmetric (R) $= R \cup \{(4,2),(3,4)\}$

Ex: Consider the relation < (less than) on the set N of positive integers. Then

reflexive $(<) = < \cup \Delta = \leq = \{(a,b): a \leq b\}$

Symetric $(<) = < \cup > = \{(a,b): a \neq b\}$

Transitive closure:-

Let R be a relation on a set A. Recall that $R^2 = R \circ R$ and $R^n = R^{n-1} \circ R$, we define

$$R^* = \bigcup_{i=1}^{\infty} R^i$$

The following thm applies.

Thm 2.4: R^* is the transitive closure of a relation R.

Suppose A is a finite set with n elements. Then we show in

$$R^* = R \cup R^2 \cup R^3 \dots \cup R^n$$

So, transitive$(R) = R \cup R^2 \cup \cdots \cup R^n$

Ex: R on $A = \{1,2,3\}$

$R = \{(1,2),(2,3),(3,3)\}$

Then $R^2 = R \circ R = \{(1,3),(2,3),(3,3)\}$ and

$R^3 = R^2 \circ R = \{(1,3),(2,3),(3,3)\}$

accordingly transitive$(R) = R \cup R^2 \cup R^3 = \{(1,2),(2,3),(3,3),(1,3)\}$

Equivalence Relations and Partitions:

⇒ A Partition P of S is collection $\{A_i\}$ of nonempty ~~set~~ subsets of S with the following two properties:

(1) Each $a \in S$ belongs to some A_i

(2) If $A_i \neq A_j$, then $A_i \cap A_j = \phi$

In other words, a partition P of S is a subdivision of S into disjoint nonempty sets.

⇒ Suppose R is an equivalence relation on a set S. ~~into disjoint nonempty sets.~~ For each a in S, let $[a]$ denote the set of elements of S ~~set~~ to which a is related under R; that is

$$[a] = \{x : (a,x) \in R\}$$

Partial ordering Relation:

A relation R on a set S is called a partial ordering or a partial order if R is reflexive, antisymmetric, and transitive. A set S with a partial ordering R is called a partially ordered set or poset.

Ex: ① The relation $\subseteq$ of set inclusion is a partial ordering ~~on the set of N of positive~~ on any collection of sets since set inclusion has the three desired properties that is.

(i) $A \subseteq A$ for any set A (ii) If $A \subseteq B$ and $B \subseteq A$ then $A = B$

(iii) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$

Ex ② $\leq$

Ex ③ "a divides b"

Equivalence Relations Example

(a) Consider the set L of lines and the set T of triangles in the Euclidean plane. The relation "is parallel to or identical to" is an equivalence relation on L, and congruence and similarity are ~~equiv~~ equivalence relations on T.

Congruent meaning in Maths :- Two shapes are congruent if they have the same shape and size. We can also say if two shapes are congruent, then the mirror image of one shape is the same as the other.

(b). The classification of animals by species, that is, the relation "is of the same species as", is ~~the~~ an equivalence relation on the set of animals.

(c). The relation $\subseteq$ of set ~~of~~ inclusion is not an equivalence relation. It is reflexive and transitive, but it is not symmetric since $A \subseteq B$ does not employ $B \subseteq A$.

(d) Let m be a fixed positive integer. Two integers a and b are said to be congruent modulo m, written

$$a \equiv b \pmod{m}$$

if m divides $a-b$.

For example, $m=4$ we have $11 \equiv 3 \pmod 4$ since 4 divides $11-3$. and $22 \equiv 6 \pmod 4$ since 4 divides $22-6$.

$\Rightarrow$ The relation of congruence of modulo m is an equivalence relation.

Note:-

$\Rightarrow$ The property of transitivity can also be expressed in terms of the composition of relations. For a relation R on A we define

$R^2 = R \circ R$ and, more generally, $R^n = R^{n-1} \circ R$

Then we have the following result.

Thm:- A relation R is transitive if and only if $R^n \subseteq R$ for $n \geq 1$

(A) Partial order :- A binary relation R on a set A is a partial order if and only if it is
(1) reflexive
(2) antisymmetric, and
(3) transitive.

→ The ordered pair <A, R> is called a poset (partially ordered set) when R is a partial order.

Ex:- The less-than-or-equal-to relation on the set of integers I is a partial order, and the set I with this relation is a poset.

(B) Total order :- A binary relation R on a set A is a total order if and only if it is
(1) a partial order, and
(2) for any pair of elements a and b of A, $\langle a,b \rangle \in R$ or $\langle b,a \rangle \in R$.

⇒ That is, every element is related with every element one way or the other.
A total order is also called a linear order.

Ex:- The less-than-or-equal-to relation on the set of integers I is a total order.
→ The strictly-less-than and proper-subset relations are not partial order because they are not reflexive. The are examples of some relation called quasi order.

(C) Quasi order :- A binary relation R on a set A is a quasi order if and only if it is → {No element is related to itself }
(1) irreflexive, and
(2) transitive.

⇒ A Quasi order is necessarily antisymmetric as one can easily verify.

Ex:- (i) less than
(ii) proper subset.

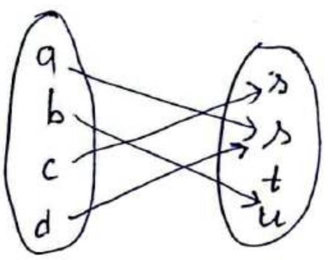
Functions

Def^n: Suppose that to each element of a set A we assign a unique element of a set B; the collection of such assignment is ~~known~~ called a function from A into B. The set A is called the domain of the function, and the set B is called the codomain.

$$\underbrace{f : A \to B}_{f \text{ is a function from } A \text{ into } B.}$$

Ex: $f(x) = x^2$ or $x \to x^2$ or $y = x^2$

f — function, x — variable; $\to$ — "goesinto"; y is dependent variable; x is independent variable.



A B

Identity function: Let A be any set. The function from A into A which assigns to each element that element itself is called the Identity function on A and is usually denoted by I_A or simply I. In other words.

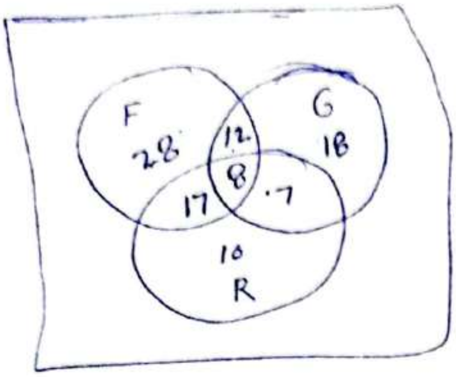
$\boxed{I_A(a) = a}$ for every element ~~a~~ in A.

Def^n: A function $f : A \to B$ is a relation from A to B (i.e., a subset of $A \times B$) such that each $a \in A$ belongs to a unique ordered pair (a, b) in f.

Ex: Consider the following relations on the set $A = \{1, 2, 3\}$:

$f = \{(1,3), (2,3), (3,1)\}$
$g = \{(1,2), (3,1)\}$
$h = \{(1,3), (2,1), (1,2), (3,1)\}$

which ~~are~~ is not ~~a~~ function



Total 120 Students

CS, AI, ML

(Hindi, English, Urdu)

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

$$= 65 + 45 + 42 - 20 - 25 - 15 + 8$$

$$= 100$$

$120 - 100 = 20$ Students ~~does~~ not study any of the language.

$20 - 8 = 12$ study F.

$28 + 18 + 10 = 56$

Let there are 120 students in Language College and three languages to Study. Hindi, Urdu and Sanskrit.

65 Study Hindi
45 Study Urdu
42 study Sanskrit
20 Study Hindi and Urdu
25 study Hindi and Sanskrit.
15 Study Urdu and Sanskrit
8 study all three languages.

How many Students are there who study only one language?

Composition functions:-

let $f: A \to B$ and $g: B \to C$; that is, when Codomain of f is the domain of g. Then we may define a new function from A to C, called the composition of f and g and written $g \circ f$.

$$(g \circ f)(a) = g(f(a))$$

Note:- Consider any function $f: A \to B$, then

$$f \circ I_A = f \quad \text{and} \quad I_B \circ f = f$$

where I_A and I_B are the Identity functions on A and B, respectively.

One to one functions:-

A function $f: A \to A$ is said to be one-to-one if different elements in the domain A have distinct images.

or

f is one to one if $f(a) = f(a')$ implies $a = a'$

Onto functions:-

A function $f: A \to B$ is ~~called~~ said to be an onto function if each element of B is the image of some element of A.

In other words, $\Rightarrow$ $f: A \to B$ is onto if the image of f is the entire codomain, i.e., if $f(A) = B$.

$\Rightarrow$ f is a function from A onto B.

or

f maps A onto B

Invertible function:- A function $f: A \to B$ is invertible if its inverse relation f^{-1} is a function from B to A.

Th^m: $\Rightarrow$ A function $f: A \to B$ is invertible if and only if f is both one-to-one and onto.

Floors and Ceiling functions

$\lfloor x \rfloor$, called the floor of x, denotes the greatest integer that does not exceed x.

$\lceil x \rceil$, call the ceiling of x, denotes the least integer that is not less than x.

If x is itself an integer then, $\lfloor x \rfloor = \lceil x \rceil$, otherwise $\lfloor x \rfloor + 1 = \lceil x \rceil$

$\lfloor 3.14 \rfloor = 3$ $\lfloor \sqrt{5} \rfloor = 2$ $\lfloor -8.5 \rfloor = -9$ $\lceil -8.5 \rceil = -8$

$\lfloor 7 \rfloor = 7$, $\lceil 7 \rceil = 7$.

Integer and Absolute Value function

⇒ The integer value of x, written INT (x), converts x into an integer by deleting (truncating) the fractional part of the number. Thus

INT $(3.14) = 3$ INT $(\sqrt{5}) = 2$ INT $(-8.5) = -8$, INT $(7) = 7$

⇒ The absolute value of the real number x, written ABS(x) or $|x|$, is defines as the greater of x or $-x$.

$|-15| = 15$, $|7| = 7$, $|-3.33| = 3.33$, $|4.44| = 4.44$.

⇒ $|x| = |-x|$ and, for $x \neq 0$, $|x|$ is positive.

Recursively defined functions:-

A function is said to be recursively defined ~~function~~ if the function definition refers to itself.

Two essential properties.

1). There must be certain arguments, called base values, for which the function does not refer to itself.

2). Each time the function does refer to itself, the argument of the function must be closer to a base value.

Ex: factorial function, fibonacci sequence.

Growth of function

Algorithms and Functions

$f(x)$

⇒ An algorithm M is a finite step by-step list of well-defined instructions for solving a particular problem, say, to find the output $f(x)$ for a given function f with input x. There may be more than one way to obtain $f(x)$. The particular choice of the algorithm M to obtain $f(x)$ may depend on the "efficiency" or "complexity" of the algorithm.

Complexity of Algorithms.

- The Complexity of an algorithm M is the function $f(n)$ which gives the running time and/or storage space requirement of the algorithm in terms of the size n of the input data

Two Cases

(i) Worst Case :- The maximum value of $f(n)$ for any possible input.

(ii) Average case :- The expected value of $f(n)$.

Rate of Growth :- Big O Notation

Suppose M is an algorithm, and suppose n is the size of the input data. Clearly the complexity $f(n)$ of M increases as n increases. It is usually the rate of increase of $f(n)$ that we want to examine.

$$\log_2 n,\ n,\ n\log_2 n,\ n^2,\ n^3,\ 2^n.$$

Def^n:. Let $f(x)$ and $g(x)$ be arbitrary functions defined on R or a subset of R. We say "$f(x)$ is of order $g(x)$", written

$$f(x) = O(g(x))$$

if there exists a real number k and a positive constant C such that, for all $x > k$, we have

$$|f(x)| \le C\,|g(x)|$$

Ex:.

$7x^2 - 9x + 4 = O(x^2)$

$8x^3 + 832x - 248 = O(x^3)$

(a) Linear search : $O(n)$
(b) Binary search : $O(\log n)$
(c) Bubble sort : $O(n^2)$
(d) Merge-sort : $O(n \log n)$

Recursive functions :-

Let a and b be positive integers, and suppose Q is defined recursively as follows:

$$Q(a,b) = \begin{cases} 0 & \text{if } a<b \\ Q(a-b,b)+1 & \text{if } b \le a \end{cases}$$

(a) find (i) $Q(2,5)$, (ii) $Q(12,5)$
(b) what does this function Q do? find $Q(5861,7)$.

Ans (a)(i) $Q(2,5) = 0$ since $2<5$

(ii) $Q(12,5) = Q(7,5)+1$
$= [Q(2,5)+1]+1 = Q(2,5)+2$
$= 0+2 = 2$

(b) Each time b is subtracted from a, the value of Q is increased by 1. Hence $Q(a,b)$ finds the quotient when a is divided by b. Thus $Q(5861,7) = 837$.

14,337
$\boxed{14,325}$

~~1000185643600~~

Mathematical Induction

Natural Numbers - 1, 2, 3, 4, ...
(+ve integers)

Principle of Mathematical Induction

Suppose there is a given statement $P(n)$ involving the natural number n such that

(i) The statement is true for $n=1$, i.e. $P(1)$ is true, and

(ii) If the statement is true for $n=k$ (where k is some positive integers); then the statement is also true for $n=k+1$, i.e., truth of $P(k)$ implies the truth of $P(k+1)$.

Then, $P(n)$ is true for all natural numbers n.

Property (i) is simply a statement of fact. There may be situations when a statement is true for all $n \geq 4$. In this case, step 1 will start from $n=4$ and we shall verify the result for $n=4$. i.e. $P(4)$.

Property (ii) is a conditional property. It does not assert that the given statement is true for $n=k$, but only that if it is true for $n=k$, then it is also true for $n=k+1$. So, to prove that the property holds, only prove the conditional proposition:

→ If the statement is true for $n=k$, then it is also true for $n=k+1$.
Inductive hypothesis.
↳ This is sometime referred to as inductive step.

Ex:
$1 = 1^2 = 1$
$4 = 2^2 = 1+3$
$9 = 3^2 = 1+3+5$
$16 = 4^2 = 1+3+5+7$ etc.

It is worth to be noted that the sum of the first two odd natural numbers is the square of second natural number, sum of the first three natural numbers is the square of third natural number and so on. Thus, from this pattern it appear that

$$1+3+5+7+ \cdots + (2n-1) = n^2$$ i.e.

the sum of the first n odd natural numbers is the square of n.

Let us write.

$$\boxed{P(n) : 1+3+5+7+ \cdots + (2n-1) = n^2}$$

we wish to prove that $P(n)$ is true for all n.

The first step.

$P(1)$ is true:-

$1 = 1^2$, i.e. $P(1)$ is true.

Inductive step : let $P(k)$ is true for some positive integer k and we need to prove that $P(k+1)$ is true. Since $P(k)$ is true.

$$1+3+5+7+ \cdots (2k-1) = k^2$$

Consider

$$1+3+5+7+ \cdots + (2k-1) + \{2(k+1)-1\}$$

$$= k^2 + (2k+1) = (k+1)^2$$

Therefore $P(k+1)$ is true and the inductive proof is completed.

Hence, $P(n)$ is true for all natural numbers.

Ex(1) for all $n \geq 1$, prove that

$$1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Sol:-

$$P(n) : 1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$P(1): 1 = \frac{1(1+1)(2\times1+1)}{6} = \frac{1\times2\times3}{6} = 1$ which is true

let for k is true

$$1^2+2^2+3^2+4^2+\cdots+k^2 = \frac{k(k+1)(2k+1)}{6}$$

we shall now prove that $P(k+1)$ is also true; NOW, we have

$$(1^2+2^2+3^2+4^2+\cdots+k^2)+(k+1)^2$$
$$= \frac{k(k+1)(2k+1)}{6}+(k+1)^2$$
$$= \frac{k(k+1)(2k+1)+6(k+1)^2}{6}$$
$$= \frac{(k+1)(2k^2+7k+6)}{6}$$
$$= \frac{(k+1)(k+1+1)\{2(k+1)+1\}}{6}$$

Thus, $P(k+1)$ is true whenever $P(k)$ is true.
Hence, from the principle of mathematical induction, the statement $P(n)$ is true for all natural numbers n.

Ex:- (i) Prove that $2^n > n$ for all positive integers n. $\rightarrow$ $\{2^k > k$ multiply 2 both $2.2^k > 2k$ $2^{k+1} > 2k = k+k > k+1$ $2^{k+1} > k+1\}$

(ii) for all $n \geq 1$, prove that

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\cdots+\frac{1}{n(n+1)} = \frac{n}{n+1}$$

(iii) for every positive integer n, prove that 7^n-3^n is divisible by 4.

(iv) Prove that $(1+x)^n \geq (1+nx)$, for all natural number n, where $x > -1$.

(v) Prove that.

$$1^2+2^2+\cdots+n^2 > \frac{n^3}{3}, n \in N$$

(v) Prove the rule of exponent $(ab)^n = a^n b^n$ by using principle of mathematical induction for every natural number.

Soln (ii)

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\cdots+\frac{1}{k(k+1)} = \frac{k}{k+1}$$

$$\underbrace{\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\cdots+\frac{1}{k(k+1)}}+\frac{1}{(k+1)(k+2)}$$

$$= \frac{k}{(k+1)}+\frac{1}{(k+1)(k+2)} = \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{k^2+2k+1}{\cdots}$$

$$\frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2} = \frac{k+1}{(k+1)+1}$$ Proved

Ans :-

$7^k - 3^k$ is divisible by 4

we can write $7^k - 3^k = 4d$, where $d \in N$.

$$7^{(k+1)} - 3^{(k+1)} = 7^{(k+1)} - 7.3^k + 7.3^k - 3^{(k+1)}$$

$$= 7(7^k - 3^k) + (7-3).3^k = 7(4d) + (7-3)3^k$$

$$= 7(4d) + 4.3^k = 4(7d + 3^k)$$

divisible by 4.

CamScanner

Proof Techniques

* The following three definitions are central to the execution of our proofs.

Definition 1: An integer number n is even if and only if there exists a number k such that $n = 2k$.

Definition 2: An integer number n is Odd if and only if there exists a number k such that $n = 2k+1$

Definition 3:. Two integers a and b are Consecutive if and only if $b = a+1$.

(A) Direct Proof [Proof by Construction]

→ In a Constructive proof one attempts to demonstrate $P \Rightarrow Q$ directly.

→ simplest and easiest method of proof.

1. Assume that P is true
2. Use P to show that Q must be true.

Theorem 1:- If a and b are Consecutive Integers, then the sum $a+b$ is odd.

Proof:- Assume that a and b are Consecutive integers. Because a and b are Consecutive we know that $b=a+1$. Thus, the sum $a+b$ may be re-written as $2a+1$. Thus, there exists a number k such that $a+b = 2k+1$ so the sum $a+b$ is odd.

(B) Proof by Contradiction

→ we Can use this to demonstrate $P \Rightarrow Q$ by assuming both P and $\neg Q$ are simultaneous true and deriving a contradiction when we derive this contradiction it means that one of our assumptions was (undefendable) untenable. Presumably we have either assumed or already proved P to be true so that finding a contradiction implies that $\neg Q$ must be false.

The method of proof by contradiction

1. Assume that P is true
2. Assume that $\neg Q$ is true
3. Use P and $\neg Q$ to demonstrate a contradiction

Theorem 2. If a and b are consecutive integers, then the sum $a+b$ is odd.

Proof:- Assume that a and b are consecutive integers. Assume also that the sum $a+b$ is not odd. Because the sum $a+b$ is not odd, there exists no number k such that $a+b=2k+1$. However, the integers a and b are consecutive integers, so we may write the sum $a+b$ as $2a+1$. Thus, we have derived that $a+b \neq 2k+1$, for any integer k and also that $a+b=2a+1$. This is a contradiction. If we hold a and b are consecutive then we know that the sum $a+b$ must be odd.

(c) Proof by Induction.

1. Show that a propositional form $P(x)$ is true for some basic case.
2. Assume that $P(n)$ is true for some n, and show that this implies that $P(n+1)$ is true.
3. Then, by the principle of induction, the propositional form $P(x)$ is true for all n greater or equal to the basic case.

Theorem 3. If a and b are consecutive integers, then the sum a+b is odd.

Proof:. Define the propositional form F(x) to be true when the sum of x. and its successor is odd.

(Step 1) consider the proposition F.(1), The sum 1+2=3 is odd because we can demonstrate there exists an integer K such that 2K+1 =3. Thus; F(x) is true when x=1.

(Step(2)): Assume that F(x) is true for some x. Thus for some x we have that x+(x+1) is odd.

→ we add 1 to both x and x+1 which gives the sum (x+1)+(x+2).

→ we claim two things: First, the sum (x+1)+(x+2)=F(x+1). second, we claim that adding two to any integers does not change that integer's evenness or oddness, with these two observations we claim that F(x) is odd implies F(x+1) is odd.

Step(3) By the principle of mathematical induction we thus claim F(x) is odd for all integers x. Thus, the sum of any two consecutive integers is odd.

(D) Proof by Contrapositive.

Two proposition are equivalent mean that. if one can prove P ⟹ Q then they have also proved ¬Q ⟹ ¬P. and vice versa.

Theorem 4:. If the sum a+b is not odd, then a and b are not consecutive integers.

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Proof:- Assume that the sum of the integers ~~and~~ ~~on~~ a and b is not odd. There, there exists no integer k such that $a+b = 2k+1$. Thus, $a+b \neq k+(k+1)$ for all integers k. Because k+1 is the successor of k, this implies that a and b cannot be consecutive integers.