Classification of PDE

1. Classify the following operators:

   (i) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial t} + 2\frac{\partial^2 u}{\partial t^2}$

   (ii) $3\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x \partial t} + 3\frac{\partial^2 u}{\partial t^2} + x\frac{\partial u}{\partial x} - t\frac{\partial u}{\partial t} + 3t$

   (iii) $4\frac{\partial^2 u}{\partial x^2} + 5\frac{\partial^2 u}{\partial x \partial t} - \frac{\partial^2 u}{\partial t^2}$

   (iv) $x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} + 2x\frac{\partial^2 u}{\partial y^2} + 2\frac{\partial u}{\partial x} + u$

   (v) $x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} + 2x\frac{\partial^2 u}{\partial y^2} + 2\frac{\partial u}{\partial x} + u$

   (vi) $u_{xx} - 3xu_{xy} - 2yu_{yy} + u_x - u_y$

   (vii) $z_{xx} - 2z_{tt}$

   (viii) $u_{xt} - tu_{tt} + x^2u_{xx}$

2. Classify the following PDE

   (i) $(1-x^2)\frac{\partial^2 u}{\partial x^2} - 2xy\frac{\partial^2 u}{\partial x \partial y} + (1-y^2)\frac{\partial^2 u}{\partial y^2} - 2u = 0$

   (ii) $y^2r - 2xys + x^2t = \frac{y^2}{z}p + \frac{x^2}{y}q$

   (iii) $\sqrt{y^2+x^2}u_{xx} + 2(x-y)u_{xy} + \sqrt{y^2+x^2}u_{yy} = 0$

   (vI) $Z_{xx} + 2xZ_{xy}(1-y^2)Z_{yy} = 0$

Method of Separation of Variables

1. Solve $4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 4u$ given that $u = 3e^{-2x} - e^{-5x}$ when $t = 0$
2. Solve $\frac{\partial^2 u}{\partial x \partial t} = e^{-t}\cos x$given that $u = 0$ when $t = 0$ and $\frac{\partial u}{\partial t} = 0$ when $x = 0$.
3. Solve the PDE $u_{xx} = u_y + 3, u(0,y) = 0, \frac{\partial}{\partial x}u(0,y) = 1 + e^{-2y}$
4. Solve $2\frac{\partial z}{\partial x} + 3\frac{\partial z}{\partial y} + 5z = 0; z(0,y) = 2e^{-y}$
5. $3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0; u(x,0) = 4e^{-x}$
6. $y^3\frac{\partial u}{\partial x} + x^2\frac{\partial u}{\partial y} = 0$
7. $x\frac{\partial^2 u}{\partial x \partial y} + 2yu = 0$
8. Solve $\frac{\partial z}{\partial x} + \frac{\partial^2 u}{\partial y^2} = 0; z(x,0) = 0, z(x,\pi) = 0, z(0,y) = 4\sin 3y$

One Dimensional Wave Equation

1. A tightly stretched string with fixed points $x = 0$ and $x = l$ is initially in a position given by $y = Asin^3 \frac{\pi x}{l}$. If it is released from rest from this position. Find the displacement $y(x, t)$.
2. A tightly stretched flexible string has its ends fixed at $\mathrm{x} = 0$ and $\mathrm{x} = \mathrm{l}$. At time $\mathrm{t} = 0$, the string is given a shape defined by $\mathrm{F(x)} = \mu\mathrm{x}(\mathrm{l} - \mathrm{x})$, μ is a constant and then released. Find the displacement $\mathrm{y(x,t)}$of any point x of the string at any time $\mathrm{t} > 0$.
3. A slightly stretched string with fixed end points $\mathrm{x} = 0$ and $\mathrm{x} = \mathrm{l}$ is initially at rest in its equilibrium position. If it is set vibrating by giving to each of its points an initial velocity $\lambda\mathrm{x}(\mathrm{l} - \mathrm{x})$, find the displacement of the string at any distance 'x' from one end at any time t.
4. If a string of length L is initially at rest in equilibrium position and each of its points is given the velocity $\left(\frac{\partial \mathrm{y}}{\partial \mathrm{t}}\right)_{\mathrm{t}=0} = \mathrm{bsin}^3 \frac{\pi \mathrm{x}}{\mathrm{L}}$. Find the displacement $\mathrm{y(x, t)}$.
5. A string is stretched between two fixed points (0, 0) and (l, 0) and released at rest from the initial deflection given by $\mathrm{f(x)} = \begin{cases} \left(\frac{2\mathrm{k}}{\mathrm{l}}\right)\mathrm{x}, & 0 < \mathrm{x} < \frac{\mathrm{l}}{2} \\ \left(\frac{2\mathrm{k}}{\mathrm{l}}\right)(\mathrm{l} - \mathrm{x}), & \frac{\mathrm{l}}{2} < \mathrm{x} < \mathrm{l} \end{cases}$
6. And find the deflection of the string at any time.
7. Solve: $\mathrm{y_{tt}} = 4\mathrm{y_{xx}}\,; \mathrm{y(0,t), y(x,0)} = 0, \left(\frac{\partial \mathrm{y}}{\partial \mathrm{t}}\right)_{\mathrm{t}=0} = 3\mathrm{sin}2\pi\mathrm{x} - 2\mathrm{sin}5\pi\mathrm{x}$.
8. A tight string of length 20 cm fastened at both sides is displaced from its position of equilibrium by imparting to each of its points an initial velocity given by $\mathrm{v} = \begin{cases} \mathrm{x}, & 0 \le \mathrm{x} \le 10 \\ 20 - \mathrm{x}, & 10 \le \mathrm{x} \le 20 \end{cases}$, x being the distance from one end. Determine the displacement at any subsequent time.
9. The vibrations of an elastic string are governed by the partial differential equation $\frac{\partial^2 \mathrm{u}}{\partial \mathrm{t}^2} = \frac{\partial^2 \mathrm{u}}{\partial \mathrm{x}^2}$. The length of the string is π and ends are fixed. The initial velocity is zero and the initial deflection is $\mathrm{u(x, 0)} = 2(\mathrm{sinx} + \mathrm{sin3x})$. Find the deflection $\mathrm{u(x, t)}$ of the vibrating string at any time t.
10. Find the deflection $u(x, t)$ of a tightly stretched vibrating string of unit length that is initially at rest and whose initial position is given by $\sin \pi x + \frac{1}{3}\sin 3\pi x + \frac{1}{5}\sin 5\pi x \;, 0 \le x \le 1$.

One Dimensional Heat Flow Equation

1. A rod of length 'l' with insulated sides in initially at a uniform temperature u_0. Its ends are suddenly cooled to 0^0 C and are kept at that temperature. Find the temperature function $u(x,t)$.
2. Find the temperature in a bar of length 2 whose ends are kept at zero and lateral surface insulated if the initial temperature is $\sin\frac{\pi x}{2} + 3\sin\frac{5\pi x}{2}$.
3. An insulated rod of length l has its ends A and B maintained at 0^0 C and 100^0 C respectively until steady state conditions prevail. If B is suddenly reduced to 0^0 C and maintained at 0^0 C, find the temperature at a distance x from A at time t.
4. The ends A and B of a rod of length 20 cm are at temperatures 30^0 C and 80^0 C until steady state prevails. Then the temperature of the rod ends is changed to 40^0C & 60^0C respectively. Find the temperature distribution function $u(x,t)$. The specific heat, dencity and the thermal conductivity of the material of the rod are such that the combination $\frac{\kappa}{\sigma\rho} = c^2 = 1$.
5. The temperature distribution in a bar of length π which is perfectly insulated at ends x=0 and x=π is governed by partial differential equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ .Assuming the initial temperature distribution as $u(x,0) = f(x) = \cos 2x$, find the temperature distribution at any instant of time.
6. Solve the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ with boundary condition

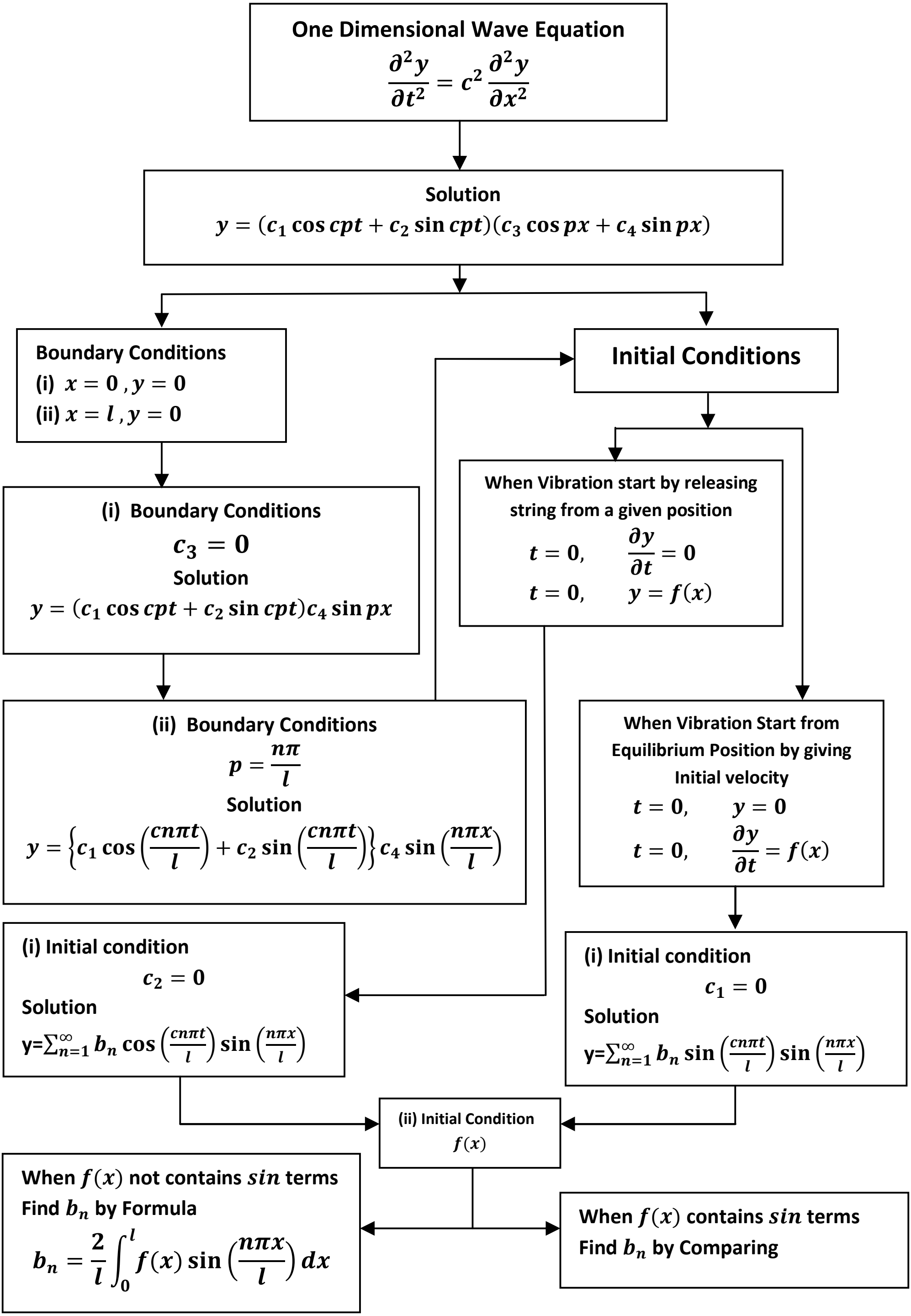
   $u(x,0) = 3\sin n\,\pi x$, $u(0,t) = 0, u(l,t) = 0$, where $0 < x < l$.
7. Solve $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ under the conditions

   (i) $u \neq \infty$ it $t \to \infty$ (ii) $\frac{\partial u}{\partial x} = 0$ for $x = 0$ and $x = l$

   (iii) $u = lx - x^2$ for $t = 0$ between $x = 0$ and $x = l$.
8. A bar with insulated sides is initially at temperature 0^0C throughout. The end $x = 0$ is kept at 0^0C, and heat is suddenly applied at the end $x = l$ so that $\frac{\partial u}{\partial x} = A$ for $x = l$, where A is a constant. Find the temperature function $u(x,t)$.
9. A homogeneous rod of conducting material of length 100 cm has its ends kept at zero temperature and the temperature initially is $u(x,0) = \begin{cases} x, & 0 \leq x \leq 50 \\ 100 - x, & 50 \leq x \leq 100 \end{cases}$.
10. Solve $u_t = a^2 u_{xx}$ under the conditions $u_x(0,t) = 0 = u_x(\pi,t)$ & $u(x,0) = x^2$ $(0 < x < \pi)$.

Two Dimensional Heat Flow Equation

1. A rectangular plate with insulated surfaces is 8 cm wide and so long compared to its width that it may considered infinite in length without introducing an appreciable error. If the temperature along one short edge $y = 0$ is given by $u(x, 0) = 100 \sin \frac{\pi x}{8}$, $0 < x < 8$ while the two long edges $x = 0$ and $x = 8$ as well as the other short edge are kept at 0^0 C, show that the steady state temperature at any point of the plate is given by $u(x, y)100e^{-\frac{\pi y}{8}} \sin \frac{\pi x}{8}$.
2. An infinitely long plane uniform plate is bounded by two parallel edges and an end at right angles to them. The breadth is π. This end is maintained at temperature u_0 at all points and the other edges are at zero temperature. Determine the temperature at any point of the plate in the steady state.
3. A rectangular plate with insulated surfaces is 10 cm wide so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature along one short edge $y = 0$ is given by $u(x, y) = \begin{cases} 20x, & 0 < x \leq 5 \\ 20(10 - x), & 5 \ll 10 \end{cases}$ and the two long edges $x = 0$ and $x = 10$ as well as other short edge are kept at 0^0 C. Find the temperature u at any point $P(x, y)$.
4. Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, with the rectangle $0 \leq x \leq a, 0 \leq y \leq b$; given that $u(x, b) = u(0, y) = 0$ and $u(x, 0) = x(a - x)$.
5. Solve the Laplace equation $u_{xx} + u_{yy} = 0$ subject to the conditions $u(x, 0) = 0, u(x, 1) = 0, u(\infty, y) = 0$ and $(0, y) = u_0$.



One Dimensional Wave Equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

Solution

$$y = (c_1 \cos cpt + c_2 \sin cpt)(c_3 \cos px + c_4 \sin px)$$

Boundary Conditions

(i) $x = 0\,, y = 0$

(ii) $x = l\,, y = 0$

(i) Boundary Conditions

$$c_3 = 0$$

Solution

$$y = (c_1 \cos cpt + c_2 \sin cpt) c_4 \sin px$$

(ii) Boundary Conditions

$$p = \frac{n\pi}{l}$$

Solution

$$y = \left\{c_1 \cos\left(\frac{cn\pi t}{l}\right) + c_2 \sin\left(\frac{cn\pi t}{l}\right)\right\} c_4 \sin\left(\frac{n\pi x}{l}\right)$$

Initial Conditions

When Vibration start by releasing string from a given position

$t = 0, \quad \frac{\partial y}{\partial t} = 0$

$t = 0, \quad y = f(x)$

When Vibration Start from Equilibrium Position by giving Initial velocity

$t = 0, \quad y = 0$

$t = 0, \quad \frac{\partial y}{\partial t} = f(x)$

(i) Initial condition

$$c_2 = 0$$

Solution

y=$\sum_{n=1}^{\infty} b_n \cos\left(\frac{cn\pi t}{l}\right) \sin\left(\frac{n\pi x}{l}\right)$

(i) Initial condition

$$c_1 = 0$$

Solution

y=$\sum_{n=1}^{\infty} b_n \sin\left(\frac{cn\pi t}{l}\right) \sin\left(\frac{n\pi x}{l}\right)$

(ii) Initial Condition

$f(x)$

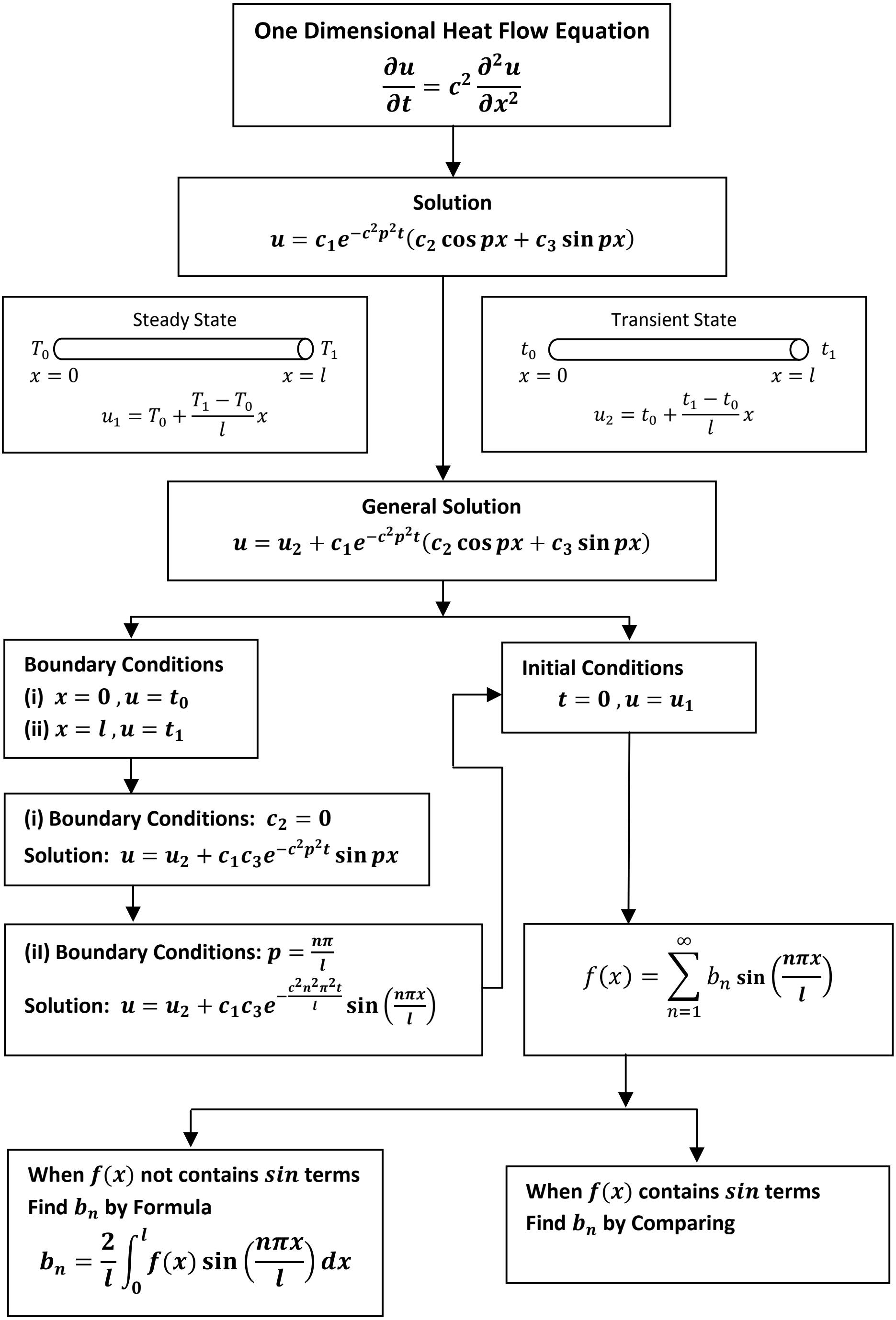
When $f(x)$ not contains sin terms

Find b_n by Formula

$$b_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

When $f(x)$ contains sin terms

Find b_n by Comparing



One Dimensional Heat Flow Equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Solution

$$u = c_1 e^{-c^2 p^2 t}(c_2 \cos px + c_3 \sin px)$$

Steady State

$$u_1 = T_0 + \frac{T_1 - T_0}{l} x$$

Transient State

$$u_2 = t_0 + \frac{t_1 - t_0}{l} x$$

General Solution

$$u = u_2 + c_1 e^{-c^2 p^2 t}(c_2 \cos px + c_3 \sin px)$$

Boundary Conditions

(i) $x = 0\,, u = t_0$

(ii) $x = l\,, u = t_1$

(i) Boundary Conditions: $c_2 = 0$

Solution: $u = u_2 + c_1 c_3 e^{-c^2 p^2 t} \sin px$

(iI) Boundary Conditions: $p = \frac{n\pi}{l}$

Solution: $u = u_2 + c_1 c_3 e^{-\frac{c^2 n^2 \pi^2 t}{l}} \sin\left(\frac{n\pi x}{l}\right)$

Initial Conditions

$t = 0\,, u = u_1$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

When $f(x)$ not contains sin terms

Find b_n by Formula

$$b_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

When $f(x)$ contains sin terms

Find b_n by Comparing